STEP MATHEMATICS 2

2021

Mark Scheme

1

$$\cos(3a + a) \equiv \cos 3a \cos a - \sin 3a \sin a$$ **M1**

$$\cos(3a - a) \equiv \cos 3a \cos a + \sin 3a \sin a$$

$$\cos 4a + \cos 2a \equiv 2 \cos 3a \cos a$$

$$\cos a \cos 3a \equiv \frac{1}{2}(\cos 4a + \cos 2a) \quad \boldsymbol{AG}$$ **A1**

$$\sin(3a + a) \equiv \sin 3a \cos a + \cos 3a \sin a$$

$$\sin(3a - a) \equiv \sin 3a \cos a - \cos 3a \sin a$$

$$\sin 4a - \sin 2a \equiv 2 \cos 3a \sin a$$

$$\sin a \cos 3a \equiv \frac{1}{2}(\sin 4a - \sin 2a)$$ **B1**

(i)

$$2 \cos 2x\,(2 \cos x \cos 3x) = 1$$

$$2 \cos 2x\,(\cos 4x + \cos 2x) = 1$$ **M1**

$$2 \cos 2x\,(2 \cos^2 2x + \cos 2x - 1) = 1$$ **M1**

$$4 \cos^3 2x + 2 \cos^2 2x - 2 \cos 2x - 1 = 0$$

$$(2 \cos^2 2x - 1)(2 \cos 2x + 1) = 0$$ **M1**

A1

Either $\cos^2 2x = \frac{1}{2}$:

$$2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$$ **A1**

Or $\cos 2x = -\frac{1}{2}$:

$$2x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$x = \frac{\pi}{3}, \frac{2\pi}{3}$$ **A1**

Therefore:

$$x = \frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$$

(ii) $$2\cos x \sin 3x \equiv \sin 4x + \sin 2x$$ **B1**

$$\tan x = \tan 2x \tan 3x \tan 4x$$ **M1**

$$\sin x \cos 2x \cos 3x \cos 4x = \cos x \sin 2x \sin 3x \sin 4x$$

$$(2\sin x \cos 3x)\cos 2x \cos 4x = (2\cos x \sin 3x)\sin 2x \sin 4x$$ **M1**

$$(\sin 4x - \sin 2x)\cos 2x \cos 4x = (\sin 4x + \sin 2x)\sin 2x \sin 4x$$

$$\sin 4x\,(\cos 2x \cos 4x - \sin 2x \sin 4x) = \sin 2x\,(\cos 2x \cos 4x + \sin 2x \sin 4x)$$

$$\sin 4x \cos 6x = \sin 2x \cos 2x$$ **M1**

$$\sin 4x \cos 6x = \frac{1}{2}\sin 4x$$ **M1**

M1

$$\sin 4x\,(2\cos 6x - 1) = 0$$ **A1**

Therefore $\cos 6x = \frac{1}{2}$ or $\sin 4x = 0$. **AG**

$\cos 6x = \frac{1}{2}$:

$$6x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}$$

$$x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}$$ **A1**

$\sin 4x = 0$:

$$4x = 0, \pi, 2\pi, 3\pi, 4\pi$$

$$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$$ **A1**

$\tan x$ is undefined at $x = \frac{\pi}{2}$ **B1**

$\tan 2x$ is undefined at $x = \frac{\pi}{4}, \frac{3\pi}{4}$ **B1**

So these are not solutions of the equation.

$$x = 0, \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \pi$$

2

(i) $$3pq - p^3 = 3(a+b)(a^2+b^2) - (a+b)^3$$ **M1**
$$= 2a^3 + 2b^3$$
$$= 2r \quad \boldsymbol{AG}$$ **A1**

(ii) $$2x^2 - 2px + (p^2 - q) = 0$$

The roots of the equation a and b satisfy: **M1**

$$a + b = p$$ **B1**
$$2ab = p^2 - q$$ **B1**
$$a^2 + b^2 = (a+b)^2 - 2ab$$ **B1**
$$= p^2 - (p^2 - q) = q$$
$$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$$
$$= p^3 - \frac{3}{2}(p^2 - q)p$$
$$= \frac{1}{2}(3pq - p^3) = r$$ **B1**

So the three equations hold. **E1**

(iii) $$a + b = s - c \ (= p)$$
$$a^2 + b^2 = t - c^2 \ (= q)$$
$$a^3 + b^3 = u - c^3 \ (= r)$$ **M1**

By part (i):

$$3(s-c)(t-c^2) - (s-c)^3 = 2(u - c^3)$$
$$3st - 3ct - 3c^2s + 3c^3 - s^3 + 3cs^2 - 3c^2s + c^3 = 2u - 2c^3$$ **M1**
$$6c^3 - 6sc^2 + 3(s^2 - t)c + 3st - s^3 - 2u = 0$$ **A1**

Therefore c is a root of the equation

$$6x^3 - 6sx^2 + 3(s^2 - t)x + 3st - s^3 - 2u = 0 \quad \boldsymbol{AG}$$ **E1**

The other roots are a and b. **B1**

The constant term is $-6 \times$ the product of the roots: **M1**

$$-6abc = 3st - s^3 - 2u$$
$$s^3 - 3st + 2u = 6v \quad \boldsymbol{AG}$$ **A1**

(iv) By (iii) a, b and c are the roots of **M1**

$$6x^3 - 18x^2 + 24x - 12 = 0$$ **A1**
$$6(x-1)(x^2 - 2x + 2) = 0$$ **M1**
$$1, 1+i, 1-i$$ **A1**

$1 + (1+i) + (1-i) = 3$

$1^2 + (1+i)^2 + (1-i)^2 = 1 + (1 + 2i - 1) + (1 - 2i - 1) = 1$

$1^3 + (1+i)^3 + (1-i)^3 = 1 + (-2+2i) + (-2-2i) = -3$ **B1**

$1(1+i)(1-i) = 2$

3

(i)	From the 1st eqn: $\lfloor x \rfloor = 4$ and $\{y\} = 0.9$	**B1**
	From the 2nd eqn: $\{x\} = 0.6$ and $\lfloor y \rfloor = -2$	**B1**
	Clear use of $x = \lfloor x \rfloor + \{x\}$ etc.	**M1**
	Solution is $x = 4.6$, $y = -1.1$	**A1**
	NB for candidates scoring *none* of the above marks, allow a **B1** for adding both eqns. to obtain $x + y = 3.5$	
(ii)	② + ③ – ①	**M1**
	$\Rightarrow\ y + \{y\} - \lfloor y \rfloor + z + \lfloor z \rfloor - \{z\} = 6.4$	
	$\Rightarrow\ 2\{y\} + 2\lfloor z \rfloor = 6.4$	**M1**
	$\Rightarrow \{y\} + \lfloor z \rfloor = 3.2$ ***AG*** or $\{x\} + \lfloor y \rfloor = 2.1$ or $\lfloor x \rfloor + \{z\} = 1.8$	**A1**
	Similar attempts at ① + ② – ③ $\Rightarrow \{x\} + \lfloor y \rfloor = 2.1$ and ① + ③ – ② $\Rightarrow \lfloor x \rfloor + \{z\} = 1.8$	**M1**
	The remaining two 2-variable eqns. correct	**A1**
	$\Rightarrow \{y\} = 0.2$ and $\lfloor z \rfloor = 3$	**B1**
	Also (respectively) $\{x\} = 0.1$ and $\lfloor y \rfloor = 2$ and $\lfloor x \rfloor = 1$ and $\{z\} = 0.8$	**B1**
	Solution is $x = 1.1$, $y = 2.2$, $z = 3.8$	**A1**
(iii)	From ② + ③ – ①, we now get $2\{y\} + \lfloor z \rfloor = 3.2$	**B1**
	From ① + ③ – ②, we still get $\lfloor x \rfloor + \{z\} = 1.8$	**B1**
	From ① + ② – ③, we now get $\{x\} + 2\lfloor y \rfloor = 2.1$	**B1**
	First solution follows immediately from (ii): namely, $x = 1.1$, $y = 1.1$, $z = 3.8$	**B1**
	For clear evidence that the second possibility exists	**M1**
	namely: $2\{y\} + \lfloor z \rfloor = 3.2 \Rightarrow \{y\} = 0.6$ and $\lfloor z \rfloor = 2$	**A1**
	and $\{x\} + 2\lfloor y \rfloor = 2.1 \Rightarrow \{x\} = 0.1$ and $\lfloor y \rfloor = 1$	**A1**
	NB $\lfloor x \rfloor = 1$ and $\{z\} = 0.8$ follows as before	
	Second solution is $x = 1.1$, $y = 1.6$, $z = 2.8$	**A1**

4

(i) $\frac{dy}{dx} = xe^x + e^x$ **M1**

Since $e^x > 0$ for all x, the only stationary point is when $x = -1$ **A1**

Coordinates of stationary point are $(-1, -\frac{1}{e})$

Sketch showing:

$y \to \infty$ as $x \to \infty$ and $y \to 0^-$ as $x \to -\infty$ **G1**

Curve passing through $(0,0)$ with stationary point at $(-1, -\frac{1}{e})$ indicated. **G1**

(ii) -1 **B1**

Sketch showing reflection of the correct portion of the graph in the line $y = x$. **G1**

domain $\left[-\frac{1}{e}, \infty\right)$ and range $[-1, \infty)$

(iii)

(a) $e^{-x} = 5x$

$xe^x = \frac{1}{5}$

M1

$f(x) = \frac{1}{5}$

Since $f(x) > 0$ there is only one solution **A1**

$x = g\left(\frac{1}{5}\right)$

(b) $2x \ln x + 1 = 0$

Let $u = \ln x$: **M1**

$$ue^u = -\frac{1}{2}$$ **M1**

The minimum value of $f(x)$ is $-\frac{1}{e}$ and $-\frac{1}{2} < -\frac{1}{e}$, so there are no solutions. **E1**

(c) $3x \ln x + 1 = 0$
Let $u = \ln x$:

$$ue^u = -\frac{1}{3}$$ **M1**

$-\frac{1}{e} < -\frac{1}{3} < 0$ so there are two solutions for u and the greater of the two will be when $u = g\left(-\frac{1}{3}\right)$. **E1**

$x = e^{g\left(-\frac{1}{3}\right)}$ is the larger value. **A1**

(d) $x = 3 \ln x$
Let $u = \ln x$:

$$ue^{-u} = \frac{1}{3}$$ **M1**

$(-u)e^{-u} = -\frac{1}{3}$, so (as in (c)) $g\left(-\frac{1}{3}\right)$ is the greater of the two possible values for $-u$. **M1**

Therefore $x = e^{-g\left(-\frac{1}{3}\right)}$ is the smaller value. **A1**

E1

(iv) $x \ln x = \ln 10$
Let $u = \ln x$:

$$ue^u = \ln 10$$ **M1**

$u = g(\ln 10)$

$$x = e^{g(\ln 10)}$$ **A1**

5

(i)

$$\frac{dy}{dx} = (x-a)\frac{du}{dx} + u$$ **M1** **A1**

$$(x-a)\left((x-a)\frac{du}{dx} + u\right) = (x-a)u - x$$

$$(x-a)^2\frac{du}{dx} = -x$$

$$u = \int \frac{-x}{(x-a)^2}dx = \int \frac{-(x-a)-a}{(x-a)^2}dx$$ **M1**

$$u = -\ln|x-a| + \frac{a}{x-a} + c$$ **A1**

$$y = -(x-a)\ln|x-a| + a + c(x-a)$$ **A1 (ft)**

(ii)

(a) The gradient of the line through $(1,t)$ and $(t,f(t))$ is $\frac{f(t)-t}{t-1} = f'(t)$ **M1**

Applying the result from (i), with a=1 or solving the d.e. directly:

$$f(x) = -(x-1)\ln|x-1| + 1 + c(x-1)$$ **B1 (ft)**

$f(0) = 0$, so $c = 1$ **B1 (ft)**

$$y = -(x-1)\ln|x-1| + x$$

$$\frac{dy}{dx} = -\ln|x-1|$$ **M1**

$-\ln|x-1| = 0$ when $x = 0$ only (since $x < 1$) and $y = 0$ **A1 (ft)**

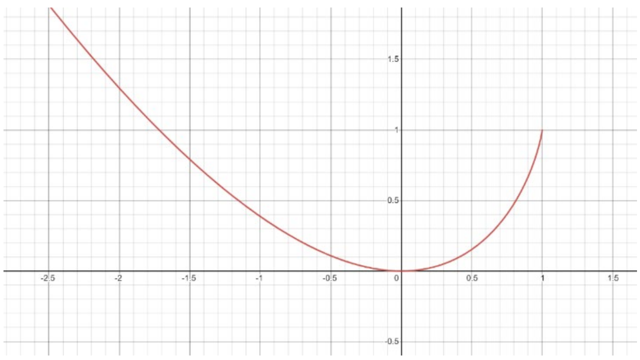
As $x \to 1^-$, $y \to 1^-$ and $\frac{dy}{dx} \to \infty$. **B1 (ft)**

Sketch showing:

Curve approaching (1,1) with a vertical tangent at that point. **G1 (ft)**

Minimum point at (0,0). **G1 (ft)**

$y \to \infty$ as $x \to -\infty$ **G1 (ft)**



(b) $f(2) = 2$, so $c = 1$ **B1 (ft)**

$$y = -(x-1)\ln|x-1| + x$$
$$\frac{dy}{dx} = -\ln|x-1|$$

$-\ln|x-1| = 0$ when $x = 2$ only (since $x > 1$) and $y = 2$. **B1 (ft)**

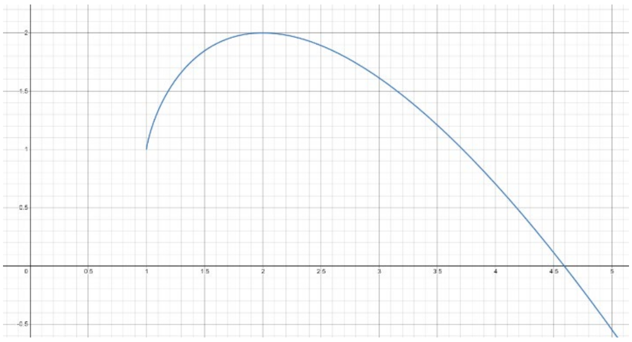
As $x \to 1^+$, $y \to 1^+$ and $\frac{dy}{dx} \to \infty$. **B1 (ft)**

Sketch showing:

Curve approaching $(1,1)$ with a vertical tangent at that point. **G1 (ft)**

Maximum point at $(2,2)$. **G1 (ft)**

The curve crossing the x-axis for some $x > 2$ and $y \to -\infty$ as $x \to \infty$ **G1 (ft)**



6

(i) The shortest distance from O to the line AB is $(R+w)\cos\alpha$ **B1**

Since $\frac{1}{3}\pi \leq \alpha \leq \frac{1}{2}\pi$, $0 \leq \cos\alpha \leq \frac{1}{2}$. **M1**

Since $w < R$, $(R+w)\cos\alpha < \frac{1}{2}(R+R) = R$, so the midpoint of the line AB lies inside the smaller circle. **E1**

(ii)

(a)
$$(R+d)^2 = (R+w)^2 + d^2 - 2d(R+w)\cos(\pi-\alpha)$$ **M1** **A1**
$$R^2 + 2Rd + d^2 = R^2 + 2Rw + w^2 + d^2 + 2d(R+w)\cos\alpha$$
$$d = \frac{w(2R+w)}{2(R-(R+w)\cos\alpha)}$$ **M1** **A1**

(b)
$$\angle O'AO = \alpha - \theta$$
$$\frac{\sin(\alpha-\theta)}{d} = \frac{\sin(\pi-\alpha)}{R+d}$$ **M1**
$$\sin(\alpha-\theta) = \frac{d\sin\alpha}{R+d}$$ **A1**

(iii)
$$\frac{d}{R} = \frac{\left(\frac{w}{R}\right)\left(2+\frac{w}{R}\right)}{2\left(1-\left(1+\frac{w}{R}\right)\cos\alpha\right)} \approx \frac{1}{1-\cos\alpha} \times \frac{w}{R}$$ **M1** **A1**

$1-\cos\alpha > \frac{1}{2}$ and $\frac{w}{R}$ is much less than 1, so $\frac{d}{R}$ is much less than 1. **E1**

$$\sin(\alpha-\theta) = \frac{\left(\frac{d}{R}\right)\sin\alpha}{1+\left(\frac{d}{R}\right)} < \frac{d}{R}$$ **M1**

$\sin(\alpha-\theta)$ is much less than 1 and so $(\alpha-\theta)$ is a small angle. **M1**

Therefore $\sin(\alpha-\theta) \approx \alpha-\theta$, so $\alpha-\theta$ is much less than 1. **E1**

(iv) The longer length is $(R+w)\times 2\alpha$

The shorter length is $(R+d)\times 2\theta$
$$S = 2\alpha(R+w) - 2\theta(R+d)$$
$$S = 2(R+d+w-d)\alpha - 2(R+d)\theta$$
$$S = 2(R+d)(\alpha-\theta) + 2(w-d)\alpha$$ **B1**

$$\alpha-\theta \approx \frac{w\sin\alpha}{R(1-\cos\alpha)}$$ **M1**
$$d-w \approx \frac{\cos\alpha}{(1-\cos\alpha)} \times \frac{w}{R}$$ **M1**

So $S \approx 2(R+d)\frac{w\sin\alpha}{R(1-\cos\alpha)} - 2\left(\frac{\cos\alpha}{(1-\cos\alpha)} \times \frac{w}{R}\right)\alpha$

As a fraction of the longer path length:
$$\frac{S}{2\alpha(R+w)} = \frac{R+d}{R+w} \times \frac{\alpha-\theta}{\alpha} + \frac{w-d}{R+w} \approx \frac{\sin\alpha}{\alpha(1-\cos\alpha)}\frac{w}{R} - \frac{\cos\alpha}{(1-\cos\alpha)}\frac{w}{R}$$ **M1**
$$S \approx \left(\frac{\sin\alpha - \alpha\cos\alpha}{\alpha(1-\cos\alpha)}\right)\frac{w}{R} \quad \textbf{AG}$$ **A1**

7

(i) $\boldsymbol{R} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ **M1**

$\boldsymbol{R} + \boldsymbol{I} = \begin{pmatrix} 1+\cos\phi & -\sin\phi \\ \sin\phi & 1+\cos\phi \end{pmatrix}$ **A1**

This must also be of the form $\begin{pmatrix} cos & -sin \\ sin & cos \end{pmatrix}$, so $(1+\cos\phi)^2 + \sin^2\phi = 1$ **M1**

$1 + 2\cos\phi = 0$

$\phi = 120°\ or\ 240°$ **A1**

In either case, three consecutive rotations is equivalent to a rotation through 0°, so

$\boldsymbol{R}^3 = \boldsymbol{I}\quad \boldsymbol{AG}$ **E1**

(ii) $\det(\boldsymbol{S}^3) = 1$

$\det(\boldsymbol{S}^3) = \det(\boldsymbol{S})^3$

Therefore $\det(\boldsymbol{S}) = 1\quad \boldsymbol{AG}$ **B1**

$\boldsymbol{S}^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} = \begin{pmatrix} a^2+bc & (a+d)b \\ (a+d)c & bc+d^2 \end{pmatrix}$

Since $\det(\boldsymbol{S}) = 1, ad - bc = 1$

$a^2 + bc = a^2 + ad - 1 = a(a+d) - 1$ **M1**

$bc + d^2 = ad + d^2 - 1 = d(a+d) - 1$

Therefore, $\boldsymbol{S}^2 = (a+d)\boldsymbol{S} - \boldsymbol{I}\quad \boldsymbol{AG}$ **A1**

$\boldsymbol{S}^3 = \boldsymbol{S}^2\boldsymbol{S} = (a+d)\boldsymbol{S}^2 - \boldsymbol{S}$ **M1**

$\boldsymbol{I} = (a+d)\big((a+d)\boldsymbol{S} - \boldsymbol{I}\big) - \boldsymbol{S}$ **M1**

$((a+d)^2 - 1)\boldsymbol{S} = (a+d+1)\boldsymbol{I}$ **A1**

If $((a+d)^2 - 1)$ and $(a+d+1)$ are non-zero, then $b = c = 0$ **B1**

In which case $ad = 1$ **M1**

since $\det(\boldsymbol{S}) = 1$ and since $\boldsymbol{S}^3 = \boldsymbol{I}$, $a = d = 1$ **A1**

But $\boldsymbol{S} \neq \boldsymbol{I}$, so this is not possible. **E1**

Therefore $a + d = -1$ **A1**

(iii) If $\boldsymbol{S} = \boldsymbol{I}$, then $\boldsymbol{S} + \boldsymbol{I} = 2\boldsymbol{I}$, which does not represent a rotation.

Therefore, the conditions of part (ii) are met and so $a + d = -1$.

Suppose that $\boldsymbol{S} + \boldsymbol{I}$ represents an anticlockwise rotation through angle θ: **M1**

$a + 1 = d + 1 = \cos\theta$

$(a+1) + (d+1) = 1$, so $a = d = -\frac{1}{2}$.

Also, $b = -c$ and $b^2 = c^2 = \frac{3}{4}$ **M1**

Therefore $\boldsymbol{S} = \begin{pmatrix} -\frac{1}{2} & \pm\frac{1}{2}\sqrt{3} \\ \mp\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}$ **A1**

Which represents a rotation through 120° or 240° **A1**

8

(i)

$$\frac{d}{dt}(t^n(1-t)^n) = nt^{n-1}(1-t)^n - nt^n(1-t)^{n-1}$$ **M1**

$$\frac{d^2}{dt^2}(t^n(1-t)^n) = n(n-1)t^{n-2}(1-t)^n - n^2t^{n-1}(1-t)^{n-1}$$

$$-n^2t^{n-1}(1-t)^{n-1} + n(n-1)t^n(1-t)^{n-2}$$ **A1**

$$= nt^{n-2}(1-t)^{n-2}[(n-1)(1-t)^2 - 2nt(1-t) + (n-1)t^2]$$ **M1**

$$= nt^{n-2}(1-t)^{n-2}[(4n-2)t^2 - (4n-2)t + (n-1)]$$

$$= nt^{n-2}(1-t)^{n-2}[(n-1) - 2(2n-1)t(1-t)] \quad \boldsymbol{AG}$$ **A1**

(ii) Integrating by parts:

$u = t^n(1-t)^n, \frac{dv}{dx} = \frac{e^t}{n!}$ **M1**

$\frac{du}{dx} = nt^{n-1}(1-t)^{n-1}(1-2t), v = \frac{e^t}{n!}$

$$T_n = \left[t^n(1-t)^n\frac{e^t}{n!}\right]_0^1 - \int_0^1 nt^{n-1}(1-t)^{n-1}(1-2t)\frac{e^t}{n!}dt$$ **M1**

$$= -\int_0^1 nt^{n-1}(1-t)^{n-1}(1-2t)\frac{e^t}{n!}dt$$

Integrating by parts:

$u = nt^{n-1}(1-t)^{n-1}(1-2t), \frac{dv}{dx} = \frac{e^t}{n!}$

$\frac{du}{dx} = nt^{n-2}(1-t)^{n-2}[(n-1) - 2(2n-1)t(1-t)], v = \frac{e^t}{n!}$

$$T_n = -\left[nt^{n-1}(1-t)^{n-1}\frac{e^t}{n!}\right]_0^1$$

$$+\int_0^1 nt^{n-2}(1-t)^{n-2}[(n-1) - 2(2n-1)t(1-t)]\frac{e^t}{n!}dt$$ **M1**

$$= \int_0^1 nt^{n-2}(1-t)^{n-2}[(n-1) - 2(2n-1)t(1-t)]\frac{e^t}{n!}dt$$

$$= \int_0^1 t^{n-2}(1-t)^{n-2}\frac{e^t}{(n-2)!} - 2(2n-1)\,t^{n-1}(1-t)^{n-1}\frac{e^t}{(n-1)!}dt$$ **M1** **M1**

$$T_n = T_{n-2} - 2(2n-1)T_{n-1} \quad for\ n \geq 2 \quad \boldsymbol{AG}$$ **A1**

(iii)

$$T_0 = \int_0^1 e^t \, dt = e - 1$$ **B1**

$$T_1 = \int_0^1 t(1-t)e^t \, dt$$

$$= \int_0^1 te^t - t^2e^t \, dt$$ **M1**

$$\int_0^1 te^t dt = [te^t]_0^1 - \int_0^1 e^t \, dt = 1$$

$$\int_0^1 t^2e^t dt = [t^2e^t]_0^1 - 2\int_0^1 te^t \, dt = e - 2$$ **M1**

$$T_1 = 1 - (e-2) = 3 - e$$ **A1**

T_0 and T_1 are both of the given form. **B1**

If T_{n-2} and T_{n-1} are both of the given form, then by part (ii):

$$a_n = a_{n-2} - 2(2n-1)a_{n-1}$$

$$b_n = b_{n-2} - 2(2n-1)b_{n-1}$$

If $a_{n-2}, a_{n-1}, b_{n-2}$ and b_{n-1} are all integers, so a_n and b_n will also be integers. **E1**

(iv) For $0 \le t \le 1$:

$0 \le t^n(1-t)^n \le 1$ **M1**

$0 \le e^t \le e$ **M1**

$0 \le \frac{t^n(1-t)^n}{n!}e^t \le \frac{e}{n!}$ and equality can only occur at t=0 or t=1, so $T_n > 0$ and is less than the area of a rectangle with width 1 and height $\frac{e}{n!}$.

$$0 < T_n < \frac{e}{n!}$$ **E1**

Therefore $a_n + b_n e \to 0$ as $n \to \infty$

Therefore $-\frac{a_n}{b_n} \to e$ as $n \to \infty$ **E1**

9

(i)

(a) The forces acting on the particle at P are:

$W = Mg$ (directed downwards) **M1**

$T_1 = m_1 g$ (directed towards Q) **A1**

$T_2 = m_2 g$ (directed towards R)

By the triangle inequality: **dM1**

$Mg < m_1 g + m_2 g$

$M < m_1 + m_2$ **A1**

$T_1^2 = T_2^2 + W^2 - 2T_2 W \cos\theta_2$

Since θ_2 is acute $\cos\theta_2 > 0$, so **M1**

$T_1^2 < T_2^2 + W^2$

$M^2 g^2 > m_1^2 g^2 - m_2^2 g^2$ **E1**

$\sqrt{m_1^2 - m_2^2} < M$

A1

(b) $QS = PS\tan\theta_1$ and $SR = PS\tan\theta_2$

If S divides QR in the ratio r: 1, then $QS = rSR$

$$r = \frac{\tan\theta_1}{\tan\theta_2}$$ **M1**

By the sine rule:

$$\frac{\sin\theta_2}{m_1 g} = \frac{\sin\theta_1}{m_2 g}$$ **M1**

By the cosine rule:

$$\cos\theta_1 = \frac{T_1^2 + W^2 - T_2^2}{2T_1 W} = \frac{m_1^2 + M^2 - m_2^2}{2m_1 M}$$ **M1**

Similarly:

$$\cos\theta_2 = \frac{m_2^2 + M^2 - m_1^2}{2m_2 M}$$ **M1**

Therefore:

$$r = \frac{\sin\theta_1}{\sin\theta_2} \times \frac{\cos\theta_2}{\cos\theta_1}$$

$$= \frac{m_2}{m_1} \times \frac{\frac{m_2^2 + M^2 - m_1^2}{2m_2 M}}{\frac{m_1^2 + M^2 - m_2^2}{2m_1 M}} = \frac{m_2^2 + M^2 - m_1^2}{m_1^2 + M^2 - m_2^2} \quad \boldsymbol{AG}$$ **dM1** **A1**

(ii) From the triangle of forces, the angle between T_1 and T_2 must be 90° (Pythagoras)

Therefore $\theta_1 + \theta_2 = 90°$ **B1**

By (i)(b)

$$r = \frac{m_2^2}{m_1^2}$$ **M1**

Let d be such that $QS = m_2^2 d$ and $SR = m_1^2 d$. **M1**

Since triangles PSQ and RSP are similar: **M1**

$$\frac{SP}{QS} = \frac{RS}{SP}$$ **M1**

$$PS^2 = m_1^2 m_2^2 d^2$$ **A1**

Therefore, $SP = m_1 m_2 d$ and QR= $(m_1^2 + m_2^2)d$, so the ratio of QR to SP is:

$$M^2 : m_1 m_2$$ **A1**

10

(i) To remain stationary relative to the train the bead would have to have horizontal acceleration a. **E1**

There is no horizontal force on the bead at the origin, so this is impossible. **E1**

(ii) When the particle is at the point (x, y):

Let the angle that the tangent to the curve makes with the horizontal be θ:

The wire is smooth, so gravity will be the only force with a component in the direction of the tangent to the curve.

The acceleration of the particle will be $\begin{pmatrix} \ddot{x} - a \\ \ddot{y} \end{pmatrix}$ **B1**

Therefore, resolving in the tangential direction: **M1**

$$m(\ddot{x} - a)cos\theta + m\ddot{y}sin\theta = -mgsin\theta$$ **A1**

A1

$$(\ddot{x} - a) + (\ddot{y} + g)tan\theta = 0$$

$$\dot{y} = \dot{x}tan\theta$$ **M1**

Therefore

$$\dot{x}(\ddot{x} - a) + (\ddot{y} + g)\dot{x}tan\theta = 0$$ **M1**

$$\dot{x}(\ddot{x} - a) + (\ddot{y} + g)\dot{y} = 0$$ **A1**

$$\frac{d}{dt}\left(\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy\right) = \dot{x}(\ddot{x} - a) + (\ddot{y} + g)\dot{y} = 0$$ **M1**

So the expression is constant during the motion. **A1**

(iii) Initially, $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy = 0$ (and throughout the motion since it is constant) **M1**

At the maximum vertical displacement $\dot{y} = 0$.

$\dot{x} = 0$ as well would only be possible at the origin (which is not maximum vertical displacement, therefore $\dot{x} = 0$ and $x \neq 0$ **M1**

Therefore, $ax = gy$

and so $g^2y^2 = a^2x^2 = a^2ky$ **M1**

Therefore, b satisfies

$$g^2b^2 = a^2kb$$

$$b = \frac{a^2k}{g^2}$$ **A1**

(iv) The square of the speed relative to the train is

$\dot{x}^2 + \dot{y}^2 = 2(ax + gy)$ **M1**

$$2\left(ax - \frac{gx^2}{k}\right)$$

$$-\frac{2g}{k}\left(x - \frac{ak}{2g}\right)^2 + \frac{a^2k}{2g}$$ **M1**

A1

Maximum speed is $a\sqrt{\frac{k}{2g}}$ **A1**

When $x = \frac{ak}{2g}$ **A1**

11

(i) $P_2 = \frac{1}{2}$ **B1**

T_3 can sit in seat S_3 if **M1**
T_1 chooses seat S_2, then T_2 chooses seat S_1
or T_1 chooses seat S_1
$P_3 = \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{2}$ **A1**

(ii) If passenger T_1 sits in seat S_k $(1 < k < n)$ then passengers T_2 to T_{k-1} all sit in their allocated seats. **E1**
The situation just before T_k arrives is then the same as for a train that did not have the $(k-1)$ seats that have been taken and for which T_k had been allocated seat S_1 **E1**

T_1 sits in seat S_1 with probability $\frac{1}{n}$, after which all the remaining passengers will get their allocated seats.

$$P(T_1 \textit{ sits in } S_1 \cap T_n \textit{ sits in } S_n) = \frac{1}{n}$$ **M1**

For $1 < k < n$, T_1 sits in seat S_k with probability $\frac{1}{n}$, so

$$P(T_1 \textit{ sits in } S_k \cap T_n \textit{ sits in } S_n) = \frac{1}{n} P_{n-k+1}$$ **M1**

If T_1 sits in S_n then it will not be possible for T_n to sit in S_n

$$P_n = \frac{1}{n} + \sum_{k=2}^{n-1} \frac{1}{n} P_{n-k+1} = \frac{1}{n}\left(1 + \sum_{r=2}^{n-1} P_r\right) \quad \textbf{\textit{AG}}$$ **A1**

(iii) $P_n = \frac{1}{2}$ **B1**
Case where $n = 1$ is shown in part (i)
Suppose $P_k = \frac{1}{2}$ for $1 \le k < n$: **M1**
$P_n = \frac{1}{n}\left(1 + (n-2) \times \frac{1}{2}\right) = \frac{1}{2}$ **A1**
Therefore, by induction $P_n = \frac{1}{2}$ **E1**

(iv) $Q_2 = \frac{1}{2}$ **B1**
For $n > 2$:
For $1 < k < n-1$:

$$P(T_{n-1} \textit{ sits in } S_{n-1} | T_1 \textit{ sits in } S_k) = Q_{n-k+1}$$ **M1**

(by similar reasoning as in part (ii))

$$P(T_{n-1} \textit{ sits in } S_{n-1} | T_1 \textit{ sits in } S_1 \textit{ or } S_n) = 1$$ **M1**

Therefore

$$Q_n = \frac{1}{n}\left(2 + \sum_{k=2}^{n-2} Q_{n-k+1}\right) = \frac{1}{n}\left(2 + \sum_{r=3}^{n-1} Q_{n-k+1}\right)$$ **A1**

Base case:
If $n = 3$, then T_2 sits in seat S_2 in any case where T_1 does not sit in seat S_2 **B1**
Suppose $Q_k = \frac{2}{3}$ for some $3 \le k < n$: **M1**
$Q_n = \frac{1}{n}\left(2 + (n-3) \times \frac{2}{3}\right) = \frac{2}{3}$ **A1**
Therefore, by induction $Q_n = \frac{2}{3}$ for $n \ge 3$ **E1**

12

(i) Player A wins the match on game n with probability $p_A(1-p_A-p_B)^{n-1}$ **B1**

The probability that A wins the match is the sum to infinity of a geometric series with **M1**
$a = p_A, r = 1 - p_A - p_B$ **M1**

$$\frac{p_A}{p_A + p_B} \quad \boldsymbol{AG}$$ **M1** **A1**

(ii) The difference between the number of games won by the two players is initially 0 and either increases or decreases by 1 after each game. **E1**

Therefore, it can only be an even number (and so the match can only be won) after an even number of games. **E1**

Considering pairs of turns at a time **M1**

The game is equivalent to that in part (i), with $p_A = p^2$ and $p_B = q^2$, **M1**

and $0 < p_A + p_B < 1$ **M1**

so the probability that A wins the match is

$$\frac{p^2}{p^2+q^2} \quad \boldsymbol{AG}$$ **A1**

(iii) Version 1:

The player has to win round 1 for the game to continue (with probability p). **M1**

Following that the game is equivalent to that in part (ii), so the probability that the player wins overall is **M1**

$$\frac{p^3}{p^2+q^2}$$ **A1**

Version 2:

The only way for the player to win is by winning two rounds in a row, so with probability **M1**

$$p^2$$ **A1**

$$p^2 - \frac{p^3}{p^2+q^2} = \frac{p^4+p^2q^2-p^3}{p^2+q^2}$$ **M1**

$$= \frac{p^4+p^2-2p^3+p^4-p^3}{p^2+q^2}$$

$$= \frac{2p^4-3p^3+p^2}{p^2+q^2}$$

$$\frac{p^2(2p-1)(p-1)}{p^2+q^2}$$ **M1**

If $1 > p > \frac{1}{2}, \frac{p^2(2p-1)(p-1)}{p^2+q^2} < 0$, so the player is more likely to win in version 1 (the cautious version) **AG** **E1**

If $0 < p < \frac{1}{2}, \frac{p^2(2p-1)(p-1)}{p^2+q^2} > 0$, so the player is more likely to win in version 2 (the bold version) **AG** **E1**